A TRACE FORMULA FOR VECTOR-VALUED MODULAR FORMS

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ABSTRACT. We present a formula for vector-valued modular forms, expressing the value of the Hilbert-polynomial of the module of holomorphic forms evaluated at specific arguments in terms of traces of representation matrices, restricting the weight distribution of the free generators.

1. INTRODUCTION

The classical theory of scalar modular forms [1, 12, 13] has been a major theme of mathematics for the last two hundred years. Its applications are numerous, ranging from number theory to topology and mathematical physics, a showpiece being the mathematics involved in the proof of Fermat's Last Theorem [4]. A major result is the description of the ring of scalar holomorphic forms as a bivariate polynomial algebra [14, 19], which allows to determine explicit bases for the spaces of holomorphic and cusp forms of different weights.

While the need to generalize the theory to vector-valued forms transforming according to some higher dimensional representation of the modular group has been recognized long ago, its systematic development has begun only recently [11, 2, 16, 3]. The importance of vector-valued modular forms for mathematics lies, besides the intrinsic interest of the subject, in the fact that important classical problems may be reduced to the study of suitable vector-valued forms, like the theory of Jacobi forms [5] or of scalar modular forms for finite index subgroups [18]; from a modern perspective, trace functions of vertex operator algebras [8, 10] satisfying suitable restrictions also provide important examples of vector-valued modular forms [20]. From the point of view of theoretical physics, vector-valued modular forms play an important role in string theory [9, 17] and two-dimensional conformal field theory [7], as the basic ingredients (chiral blocks) of torus partition functions and other correlators.

The above connections justify amply the interest in obtaining a better understanding of the spaces of vector-valued modular forms. In this respect, a most interesting question is to find explicit expressions

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for the quantities characterizing these spaces, e.g. the dimension of holomorphic or cusp forms, in terms of the associated representation of the modular group. Such results are known for some restricted class of representations, e.g. those having finite image, but their generalization is not obvious.

A result of Marks and Mason [15] states that, for a broad class of representations, the set of all holomorphic forms (of all possible weights) is a free module over the ring of scalar holomorphic forms, whose rank equals the dimension of the representation. This means that, should one know explicitly a free generating set, one would have complete control over forms for the given representation; of course, the difficulty lies in obtaining a set of free generators from the sole knowledge of the representation. More restricted, but still very useful and non-trivial information is contained in the weight distribution of the free generators, which can be encoded in the Hilbert-polynomial of the module of holomorphic forms, and whose knowledge is enough to determine, in particular, the dimensions of the spaces of holomorphic forms of different weights. The aim of the present paper is to show how one can relate the value of the Hilbert-polynomial, evaluated at specific arguments, to traces of representation operators: this restricts to a great extent the weight distribution of the free generators. The basic idea is to investigate weakly-holomorphic forms alongside holomorphic ones, leading to an explicit expression for the determinant of the matrix formed from a set of free generators, and to use the weight-shifting map to compare such determinants.

2. Scalar modular forms

A (scalar) modular form of weight w is a complex-valued function $f:\mathbf{H}\rightarrow\mathbb{C}$ that is holomorphic everywhere in the upper half-plane $\mathbf{H}=\{\tau\,|\,\mathrm{Im}\tau>0\}$, and transforms according to the rule

$$f\left(\frac{a\tau+b}{c\tau+d}\right)=(c\tau+d)^{w}\,f(\tau) \tag{1}$$

for $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in\mathbf{\Gamma}=\mathrm{SL}_2(\mathbb{Z})$. Note that the weight w should be an even integer for non-trivial forms to exist. A form $f(\tau)$ is called weakly holomorphic if it has at worst finite order poles in the limit $\tau\rightarrow\mathrm{i}\infty$, i.e. its Laurent-expansion in terms of the local uniformizing parameter $q=\exp(2\pi\mathrm{i}\tau)$ has only finitely many terms with negative exponents; it is holomorphic, respectively a cusp form, if it is bounded (resp. vanishes) as $\tau\rightarrow\mathrm{i}\infty$, meaning that its Laurent-expansion contains only non-negative (resp. positive) powers of q. For non-trivial holomorphic (resp. cusp) forms to exist the weight should be non-negative (resp. positive). We'll denote by $\mathcal{M}_{2k}$ the (infinite dimensional) linear space of weakly holomorphic forms of weight $2k$, and by M_{2k} (resp. S_{2k}) the

finite dimensional subspaces of holomorphic and cusp forms; clearly, we have the inclusions $\mathsf{S}_{2k} < \mathsf{M}_{2k} < \mathcal{M}_{2k}$. Since the product of (weakly) holomorphic (resp. cusp) forms is again a (weakly) holomorphic (resp. cusp) form, the direct sums $\mathcal{M} = \bigoplus_{k\in\mathbb{Z}} \mathcal{M}_{2k}$ and $\mathsf{M} = \bigoplus_{k=0}^{\infty} \mathsf{M}_{2k}$ (resp. $\mathsf{S} = \bigoplus_{k=1}^{\infty} \mathsf{S}_{2k}$) are graded rings.

By a well known result [1, 19], M_0 consists of constants, M_2 is empty, while M_4 and M_6, each having dimension 1, are spanned by the Eisenstein series

$$E_4(q) = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n)\, q^n \tag{2}$$

and

$$E_6(q) = 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n)\, q^n\,, \tag{3}$$

where $\sigma_k(n) = \sum_{d|n} d^k$ is the k^{th} power sum of the divisors of n. What is more, any holomorphic form may be expressed uniquely as a bivariate polynomial in the Eisenstein series $E_4(q)$ and $E_6(q)$, in other words

$$\mathsf{M} = \bigoplus_{k=0}^{\infty} \mathsf{M}_{2k} = \mathbb{C}[E_4, E_6] \tag{4}$$

as graded rings. On the other hand, there are no cusp forms of weight less than 12, while S_{12} is spanned by the discriminant form

$$\Delta(q) = \frac{1}{1728}\left(E_4(q)^3 - E_6(q)^2\right) = q \prod_{n=1}^{\infty} (1-q^n)^{24}\,, \tag{5}$$

and any cusp form of weight $k \geq 12$ is the product of $\Delta(q)$ with a holomorphic form of weight $k-12$, i.e. S is the principal ideal of M generated by $\Delta(q)$.

We shall need the following result.

Lemma 1.

$$\mathsf{M}_{2n} \subseteq E_4^{n_3} E_6^{n_2} \mathsf{M}_{12 n_\infty}$$

for any non-negative integer n, where n_k denotes the (non-negative) remainder of division of $-n$ by the integer k, and

$$n_\infty = \frac{n}{6} - \frac{n_2}{2} - \frac{n_3}{3}\,. \tag{6}$$

Proof. Since $\mathsf{M} = \mathbb{C}[E_4, E_6]$, any $f \in \mathsf{M}_{2n}$ may be written as

$$f = \sum_{a,b\geq 0} f(a,b)\, E_4^a E_6^b$$

for suitable coefficients $f(a,b) \in \mathbb{C}$ that vanish unless $2a+3b=n$, taking into account the relevant weights. This last condition can be satisfied only if $a \equiv n_3 \pmod 3$ and $b \equiv n_2 \pmod 2$. Now, $0 \leq n_k < k$ for $k>0$,

and because both exponents a and b are non-negative integers, it does follow that $a \geq n_3$ and $b \geq n_2$, leading to the conclusion that both $E_4^{n_3}$ and $E_6^{n_2}$ divide f. Since f has weight $2n$, the assertion follows. $\square$

Let's now turn to weakly holomorphic forms. According to a classic result [1, 12, 14], the ring $\mathcal{M}_0$ of scalar weakly-holomorphic forms of weight 0 is a univariate polynomial algebra generated by the Hauptmodul

$$J(q) = \frac{E_4(q)^3}{\Delta(q)} - 744 = q^{-1} + 196884q + \cdots \,, \tag{7}$$

i.e. $\mathcal{M}_0 = \mathbb{C}[J]$.

Lemma 2. *For an integer n, the module $\mathcal{M}_{2n}$ of weakly holomorphic scalar forms of weight $2n$ is generated over $\mathcal{M}_0$ by the form*

$$\mathfrak{f}_n(q) = E_4(q)^{n_3}\, E_6(q)^{n_2}\, \Delta(q)^{n_\infty} \quad , \tag{8}$$

i.e.

$$\mathcal{M}_{2n} = \mathfrak{f}_n \mathcal{M}_0 \ . \tag{9}$$

Proof. Let g denote a weakly holomorphic scalar form of weight $2n$, and let $k \geq 0$ denote its valence, i.e. the order of its pole at $q = 0$. Then the product $f = \Delta^k g$ is a holomorphic scalar form of weight $12k+2n$, and Lemma 1 applies, i.e. there exists is a holomorphic form $F \in \mathsf{M}_{12(k+n_\infty)}$ such that

$$f = E_4^{n_3} E_6^{n_2} F \,,$$

and thus

$$g = \Delta^{-k} E_4^{n_3} E_6^{n_2} F = \mathfrak{f}_n \Delta^{-n_\infty - k} F \,.$$

But the product $\Delta^{-n_\infty - k} F$ is a weight 0 weakly holomorphic scalar form, proving the assertion. $\square$

As a consequence

$$\mathcal{M} = \mathbb{C}\left[J, \mathfrak{f}_1\right] \,,$$

as a graded ring. Actually, the forms $\mathfrak{f}_n(\tau) \in \mathcal{M}_{2n}$ are holomorphic for $n > 1$, and satisfy $\mathfrak{f}_{n+6}(q) = \Delta(q)\, \mathfrak{f}_n(q)$.

Lemma 3.

$$\frac{\mathfrak{f}_n \mathfrak{f}_m}{\mathfrak{f}_{n+m}} = \left(\frac{E_4^3}{\Delta}\right)^{\mathfrak{s}_3(n,m)} \left(\frac{E_6^2}{\Delta}\right)^{\mathfrak{s}_2(n,m)} = (J+744)^{\mathfrak{s}_3(n,m)} (J-984)^{\mathfrak{s}_2(n,m)} \,, \tag{10}$$

where

$$\mathfrak{s}_k(n,m) = \begin{cases} 1 & \text{if } n_k + m_k \geq k, \\ 0 & \text{otherwise} \,. \end{cases} \tag{11}$$

Proof. From the definition Eq.(8),

$$\frac{\mathfrak{f}_n\mathfrak{f}_m}{\mathfrak{f}_{n+m}} = E_4^{n_3+m_3-(n+m)_3} E_6^{n_2+m_2-(n+m)_2} \Delta^{n_\infty+m_\infty-(n+m)_\infty} .$$

But

$$n_\infty+m_\infty-(n+m)_\infty = \frac{(n+m)_2-n_2-m_2}{2} + \frac{(n+m)_3-n_3-m_3}{3} ,$$

so the result follows by taking into account that, for $k>0$

$$n_k + m_k - (n+m)_k = k\mathfrak{s}_k(n,m) \ . \tag{12}$$

□

We conclude this section with the following counting result:

Lemma 4. *Let X denote a finite subset of $\mathbb{Z}$, k a positive integer and $0<p<k$. Then the number of elements $x\in X$ congruent to p modulo k is given by*

$$\left|\{x\in X \mid x\in k\mathbb{Z}+p\}\right| = \sum_{x\in X}\mathfrak{s}_k(p,-x) - \sum_{x\in X}\mathfrak{s}_k(p+1,-x) \ .$$

Proof. First, let's note that $x\in k\mathbb{Z}+p$ is equivalent to $(-x)_k=p$. From Eq.(11),

$$\sum_{x\in X}\mathfrak{s}_k(n,-x) = \left|\{x\in X \mid n_k+(-x)_k\geq k\}\right|$$

for any $n\in\mathbb{Z}$; but for $0<p<k$ one has $p_k=k-p$, hence

$$\sum_{x\in X}\mathfrak{s}_k(p,-x) = \left|\{x\in X \mid \ (-x)_k\geq p\}\right| \ .$$

It follows that

$$\sum_{x\in X}\left(\mathfrak{s}_k(p,-x) - \mathfrak{s}_k(p+1,-x)\right) = \left|\{x\in X \mid \ (-x)_k=p\}\right| \ .$$

□

3. Vector-valued modular forms

Let $\rho:\mathbf{\Gamma}\to\mathrm{GL}(V)$ denote a representation of $\mathbf{\Gamma}=\mathrm{SL}_2(\mathbb{Z})$ on the finite dimensional linear space V, and let n be an integer. A vector-valued modular form of weight n with multiplier ρ is a map $\mathbb{X}:\mathbf{H}\to V$ that is holomorphic everywhere in the upper half-plane $\mathbf{H}=\{\tau \mid \mathrm{Im}\tau>0\}$, and transforms according to the rule

$$\mathbb{X}\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^n \rho\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mathbb{X}(\tau) \tag{13}$$

for $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\in\mathbf{\Gamma}$. One recovers the classical notion of scalar modular forms when $\rho=\rho_0$ is the trivial (identity) representation.

As in the scalar case, a form is weakly holomorphic if it has at worst finite order poles in the limit $\tau\to\mathrm{i}\infty$, i.e. its Puiseux-expansion

in terms of the local uniformizing parameter $q=\exp(2\pi\mathrm{i}\tau)$ contains only finitely many negative powers of q; it is holomorphic, respectively a cusp form if it is bounded (resp. vanishes) as $\tau\to\mathrm{i}\infty$, meaning that its Puisseux-expansion contains only non-negative (resp. positive) powers of q. We'll denote by $\mathcal{M}_n(\rho)$ the linear space of weakly holomorphic forms of weight n, and by $\mathsf{M}_n(\rho)$ and $\mathsf{S}_n(\rho)$ the subspaces of holomorphic and cusp forms; clearly, we have the inclusions $\mathsf{S}_n(\rho)<\mathsf{M}_n(\rho)<\mathcal{M}_n(\rho)$. An obvious but important observation is that

$$\mathcal{M}_n(\rho_1\oplus\rho_2)=\mathcal{M}_n(\rho_1)\oplus\mathcal{M}_n(\rho_2) \tag{14}$$

for any two representations ρ_1 and ρ_2, and a similar decomposition holds for the spaces of holomorphic and cusp forms, allowing to reduce the general theory to the case of indecomposable representations.

We'll call a representation $\rho:\mathbf{\Gamma}\to\mathrm{GL}(V)$ even in case $\rho\begin{pmatrix}-1&0\\0&-1\end{pmatrix}=\mathbf{id}_V$, and odd if $\rho\begin{pmatrix}-1&0\\0&-1\end{pmatrix}=-\mathbf{id}_V$. Any representation ρ may be decomposed uniquely into a direct sum $\rho=\rho_+\oplus\rho_-$ of even and odd representations, and any indecomposable (in particular, any irreducible) representation is either even or odd. Combining this result with Eq.(14), one gets that it is enough to treat separately purely even and odd representations, as the general case can be reduced to these. Note that it follows from Eq.(13) that for an even (resp. odd) representation ρ there are no nontrivial forms of odd (resp. even) weight.

Since the product of a weakly-holomorphic form with a scalar form $f(\tau)\in\mathcal{M}$ is again weakly-holomorphic, it does follow that the direct sum

$$\mathcal{M}(\rho)=\bigoplus_{n\in\mathbb{Z}}\mathcal{M}_n(\rho) \tag{15}$$

is a graded module over $\mathcal{M}$. Similarly, since multiplying a holomorphic form $\mathbb{X}(\tau)\in\mathsf{M}_n(\rho)$ with a scalar holomorphic form $f(\tau)\in\mathsf{M}_{2k}$ results in a new holomorphic form $f(\tau)\mathbb{X}(\tau)\in\mathsf{M}_{n+2k}(\rho)$, and the same is true for cusp forms, the direct sums $\mathsf{M}(\rho)=\oplus_n\mathsf{M}_n(\rho)$ and $\mathsf{S}(\rho)=\oplus_n\mathsf{S}_n(\rho)$ are (graded) modules over the ring M of holomorphic scalar modular forms. An important result of Marks and Mason [15] states that $\mathsf{M}(\rho)$ is a free module of rank $d=\dim\rho$ for a broad class of representations ρ. An interesting question in this respect is to determine the distribution of the fundamental weights (i.e. the weights of a set of free generators), which may be answered by considering the Hilbert-Poincarï¿œ series $\mathfrak{M}_\rho(z)=\sum_n\dim\mathsf{M}_n(\rho)\,z^n$ of this module [6]: the number of independent free generators of weight k equals the coefficient of z^k in the Hilbert polynomial $P_\rho(z)=(1-z^4)(1-z^6)\,\mathfrak{M}_\rho(z)$.

Because the discriminant form $\Delta(\tau)$ does not vanish on the upper half-plane [1], its 12th root $\delta(\tau)=q^{1/12}\prod_{n=1}(1-q^n)^2$ (the square of Dedekind's eta function) is well-defined and holomorphic on $\mathbf{H}$, with an algebraic branch point at the cusp $\tau=\mathrm{i}\infty$. Moreover, $\delta(\tau)$ is a weight

1 cusp form with multiplier $\varkappa$, where $\varkappa$ denotes the one dimensional representation of $\mathrm{SL}_2(\mathbb{Z})$ for which[1]

$$\begin{aligned} \varkappa\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} &= -\mathsf{i} \\ \varkappa\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} &= \exp\left(\frac{4\pi\mathsf{i}}{3}\right) . \end{aligned} \tag{16}$$

It does follow that, for any representation ρ and any form $\mathbb{X}\in\mathcal{M}_n(\rho)$, one has $\delta(\tau)^k\mathbb{X}(\tau)\in\mathcal{M}_{n+k}\left(\rho\otimes\varkappa^k\right)$ for all integers $k\in\mathbb{Z}$; in other words, one has a graded bijective map

$$\begin{aligned} \boldsymbol{\varpi}_k : \mathcal{M}(\rho) &\to \mathcal{M}\left(\rho\otimes\varkappa^k\right) \\ \mathbb{X}(\tau) &\mapsto \delta(\tau)^k\mathbb{X}(\tau) . \end{aligned} \tag{17}$$

The map $\boldsymbol{\varpi}_k$ relates forms of different weights with a slightly different multiplier. Note that, since $\delta(\tau)\in\mathsf{S}_1(\varkappa)$ is a cusp form, multiplication by a positive power of $\delta(\tau)$ takes a holomorphic form into a cusp form, i.e. $\boldsymbol{\varpi}_k(\mathsf{M}_n(\rho)) < \mathsf{S}_{n+k}\left(\rho\otimes\varkappa^k\right)$ for $k>0$. The bijectivity of the weight-shifting map $\boldsymbol{\varpi}_k$ allows to reduce to a great extent the study of forms of arbitrary weights to that of forms of weight 0 (for a slightly different representation).

Let's now turn our attention to the properties of $\mathcal{M}_n(\rho)$: recall that, according to the parity of ρ, n has to be even or odd for this space to be non-trivial. The basic observation is that the product $f(\tau)\,\mathbb{X}(\tau)$ of a weakly holomorphic form $\mathbb{X}(\tau)\in\mathcal{M}_n(\rho)$ with a weakly holomorphic scalar form $f(\tau)\in\mathcal{M}_0$ of weight 0 is again a weakly holomorphic form belonging to $\mathcal{M}_n(\rho)$: in other words, $\mathcal{M}_n(\rho)$ is an $\mathcal{M}_0$-module, that can be shown to be torsion free. Taking into account the fact that $\mathcal{M}_0$ is the univariate polynomial algebra $\mathbb{C}[J]$ generated by the Hauptmodul, this means that actually $\mathcal{M}_n(\rho)$ is a free module [6], whose rank may be shown to equal the dimension d of the representation ρ. This means that there exists forms $\mathbb{X}_1,\ldots,\mathbb{X}_d\in\mathcal{M}_n(\rho)$ that freely generate $\mathcal{M}_n(\rho)$ as an $\mathcal{M}_0$-module, i.e. any weakly holomorphic form $\mathbb{X}\in\mathcal{M}_n(\rho)$ may be decomposed uniquely into a sum

$$\mathbb{X}(\tau) = \sum_{i=1}^{d} \wp_i(\tau)\,\mathbb{X}_i(\tau) , \tag{18}$$

where the coefficients $\wp_1,\ldots,\wp_d\in\mathcal{M}_0$ are weight 0 weakly holomorphic scalar forms, i.e. univariate polynomials in the Hauptmodul $J(\tau)$.

Given a free generating set $\mathbb{X}_1,\ldots,\mathbb{X}_d$ of $\mathcal{M}_n(\rho)$ over $\mathcal{M}_0$, the exterior product $\mathbb{X}_1\wedge\mathbb{X}_2\wedge\ldots\wedge\mathbb{X}_d$ (the determinant of the matrix whose columns are the $\mathbb{X}_i$) is clearly a weakly holomorphic form of weight nd

[1] $\varkappa$ generates the group of linear characters of $\mathrm{SL}_2(\mathbb{Z})$, which is cyclic of order 12; moreover, $\varkappa$ is an odd representation, and tensoring with $\varkappa$ takes an even representation into an odd one and *vice versa.*

transforming according to the one-dimensional determinant representation $\wedge^d\rho$ (the d-th exterior power of ρ): let $\mathit{\Delta}_n(\rho)$ denote the quotient of this exterior product by the coefficient of the lowest power of q in its q-expansion. Since the exterior products of different freely generating sets are proportional, it follows that $\mathit{\Delta}_n(\rho)$ is a well-defined element of $\mathcal{M}_{nd}(\wedge^d\rho)$.

Lemma 5. *For each integer n*

$$\mathit{\Delta}_n(\rho)=\delta(\tau)^{nd}\mathit{\Delta}_0\left(\rho\otimes\varkappa^{-n}\right) \ . \tag{19}$$

Proof. Since the weight-shifting map ϖ_{-n} is bijective, given a free generating set $\mathbb{X}_1,\ldots,\mathbb{X}_d$ of $\mathcal{M}_n(\rho)$, the set $\varpi_{-n}(\mathbb{X}_1),\ldots,\varpi_{-n}(\mathbb{X}_d)$ freely generates $\mathcal{M}_0(\rho\otimes\varkappa^{-n})$, hence its exterior product is proportional to both $\mathit{\Delta}_0(\rho\otimes\varkappa^{-n})$ and to $\delta(\tau)^{-nd}\mathit{\Delta}_n(\rho)$; since the leading coefficient of both expressions is 1, the two expressions should be equal. ☐

To conclude this section, we cite the following result from [3].

Proposition. *For an even representation $\rho\!:\!\mathbf{\Gamma}\to\mathrm{GL}(V)$, one has*

$$\mathit{\Delta}_0(\rho) = \left(\frac{E_4(q)}{\delta(q)^4}\right)^{\beta_1+2\beta_2}\left(\frac{E_6(q)}{\delta(q)^6}\right)^{\alpha} , \tag{20}$$

where α denotes the multiplicity of -1 as an eigenvalue of $\rho\left(\begin{smallmatrix}0 & -1\\ 1 & 0\end{smallmatrix}\right)$, while β_1 and β_2 denote the multiplicities of $\exp\left(\frac{2\pi\mathrm{i}}{3}\right)$ and $\exp\left(\frac{4\pi\mathrm{i}}{3}\right)$ as eigenvalues of $\rho\left(\begin{smallmatrix}0 & -1\\ 1 & -1\end{smallmatrix}\right)$.

Note that the eigenvalue multiplicities α, β_1 and β_2 can be determined through the relations

$$\begin{aligned}\operatorname{Tr}\rho\left(\begin{smallmatrix}0 & -1\\ 1 & 0\end{smallmatrix}\right) &= d-2\alpha\,,\\ \operatorname{Tr}\rho\left(\begin{smallmatrix}0 & -1\\ 1 & -1\end{smallmatrix}\right) &= d-\frac{3}{2}\left(\beta_1+\beta_2\right)+\mathrm{i}\frac{\sqrt{3}}{2}\left(\beta_1-\beta_2\right)\ .\end{aligned} \tag{21}$$

4. The trace formula

From now on, we shall assume that $\mathsf{M}(\rho)=\oplus_k\mathsf{M}_k(\rho)$ is a free module of rank d over the ring $\mathsf{M}=\mathbb{C}[E_4,E_6]$ of scalar holomorphic forms[2], and that $\rho\colon\mathbf{\Gamma}\to\mathrm{GL}(V)$ satisfies

$$\rho\left(\begin{smallmatrix}-1 & 0\\ 0 & -1\end{smallmatrix}\right)=(-1)^{\epsilon}\,\mathbf{id}_V\,, \tag{22}$$

where $\epsilon=0$ or 1 according to whether the representation ρ is even or odd: note that the representation $\dot\rho=\rho\otimes\varkappa^{-\epsilon}$ is always even.

[2]According to the result of Marks and Mason mentioned previously [15], this holds under rather mild conditions on ρ.

Theorem 1. *Let $F_1,\dots,F_d\in\mathsf{M}(\rho)$ denote a set of holomorphic forms of respective weights $w_1,\dots,w_d$, that generate freely the module $\mathsf{M}(\rho)$. Then the forms $\mathfrak{f}_{n-k_1}F_1,\dots,\mathfrak{f}_{n-k_d}F_d$ freely generate (over $\mathcal{M}_0$) the module $\mathcal{M}_{2n+\epsilon}(\rho)$ of weakly holomorphic forms of weight $2n+\epsilon$, where*

$$k_i=\frac{w_i-\epsilon}{2}=\left[\frac{w_i}{2}\right].$$

Proof. Let $\mathbb{X}\in\mathcal{M}_{2n+\epsilon}(\rho)$ denote a weakly holomorphic form of weight $2n+\epsilon$. Then, for some integer z large enough, the product $\Delta(\tau)^z\,\mathbb{X}(\tau)$ is holomorphic (the smallest such integer is the valence of $\mathbb{X}$), consequently there exist holomorphic scalar forms $x_1,\dots,x_d\in\mathsf{M}$ such that

$$\Delta(\tau)^z\,\mathbb{X}(\tau)=\sum_{i=1}^d x_iF_i\,,$$

with each x_i having weight $12z+2n-2k_i$. But each product $\Delta^{-z}x_i$ is a weakly holomorphic scalar form of weight $2(n-k_i)$, hence

$$\Delta^{-z}x_i=\mathfrak{f}_{n-k_i}y_i$$

for some forms $y_i\in\mathcal{M}_0$, according to Lemma 2. As a consequence,

$$\mathbb{X}=\sum_{i=1}^d y_i\left(\mathfrak{f}_{n-k_i}F_i\right)\,,$$

proving that $\mathfrak{f}_{n-k_1}F_1,\dots,\mathfrak{f}_{n-k_d}F_d$ generate $\mathcal{M}_{2n+\epsilon}(\rho)$; that they are free generators is obvious, since any relation between them would lead (after multiplication by a suitable power of the discriminant form) to a relation between the generators $F_1,\dots,F_d\in\mathsf{M}(\rho)$. □

Lemma 6.

$$\prod_{i=1}^d\frac{\mathfrak{f}_{n-k_i}}{\mathfrak{f}_{-k_i}}=\frac{\mathit{\Delta}_{2n+\epsilon}(\rho)}{\mathit{\Delta}_{\epsilon}(\rho)}\,. \tag{23}$$

Proof. By Theorem 1, the forms $\mathfrak{f}_{n-k_1}F_1,\dots,\mathfrak{f}_{n-k_d}F_d$ freely generate $\mathcal{M}_{2n+\epsilon}(\rho)$, hence their exterior product is proportional to $\mathit{\Delta}_{2n+\epsilon}(\rho)$:

$$\mathit{\Delta}_{2n+\epsilon}(\rho)=\frac{1}{K}\left(\prod_{i=1}^d\mathfrak{f}_{n-k_i}\right)\left(F_1\wedge F_2\wedge\ldots\wedge F_d\right)\,, \tag{24}$$

for some nonzero constant of proportionality $K\in\mathbb{C}$ (whose precise value depends on the choice of the generating set). The important point is that this constant K equals the coefficient of the lowest power in the q-expansion of the exterior product $F_1\wedge F_2\wedge\ldots\wedge F_d$, hence it is the same for all n, and the result follows. □

Theorem 2.

$$
\begin{aligned}
|\{i\,|\,k_i\in 2\mathbb{Z}+1\}| &= \alpha\left(\dot{\rho}\right)\ ,\\
|\{i\,|\,k_i\in 3\mathbb{Z}+1\}| &= \beta_1(\dot{\rho})\ ,\\
|\{i\,|\,k_i\in 3\mathbb{Z}+2\}| &= \beta_2(\dot{\rho})\ .
\end{aligned}
\tag{25}
$$

Proof. According to Eq.(20),

$$
\frac{\Delta_0(\dot{\rho}\otimes\varkappa^{-2n})}{\Delta_0(\dot{\rho})}=\left(\frac{E_4(\tau)}{\mathfrak{d}(\tau)^4}\right)^{B_n}\left(\frac{E_6(\tau)}{\mathfrak{d}(\tau)^6}\right)^{A_n}\ ,
$$

where $B_n = \beta_1(\dot{\rho}\otimes\varkappa^{-2n})-\beta_1(\dot{\rho})+2\beta_2(\dot{\rho}\otimes\varkappa^{-2n})-2\beta_2(\dot{\rho})$ and $A_n = \alpha(\dot{\rho}\otimes\varkappa^{-2n})-\alpha(\dot{\rho})$. On the other hand, it follows from Eqs.(23) and (19) that

$$
\begin{aligned}
\frac{\Delta_0(\dot{\rho}\otimes\varkappa^{-2n})}{\Delta_0(\dot{\rho})} &= \mathfrak{d}(\tau)^{-2nd}\frac{\Delta_{2n+\epsilon}(\rho)}{\Delta_{\epsilon}(\rho)}=\mathfrak{d}(\tau)^{-2nd}\prod_i\frac{\mathfrak{f}_{n-k_i}(\tau)}{\mathfrak{f}_{-k_i}(\tau)}\\
&= \left(\frac{\mathfrak{f}_n(\tau)}{\mathfrak{d}(\tau)^{2n}}\right)^d\prod_i\frac{\mathfrak{f}_{n-k_i}(\tau)}{\mathfrak{f}_n(\tau)\,\mathfrak{f}_{-k_i}(\tau)}\\
&= \left(\frac{E_4(\tau)}{\mathfrak{d}(\tau)^4}\right)^{n_3d-3\sum_i\mathfrak{s}_3(n,-k_i)}\left(\frac{E_6(\tau)}{\mathfrak{d}(\tau)^6}\right)^{n_2d-2\sum_i\mathfrak{s}_2(n,-k_i)}\ ,
\end{aligned}
$$

according to Eq.(10). Comparing powers of E_4 and E_6 on both sides, one concludes that

$$
\begin{aligned}
3\sum_i\mathfrak{s}_3(n,-k_i) &= n_3d-B_n\ ,\\
2\sum_i\mathfrak{s}_2(n,-k_i) &= n_2d-A_n\,.
\end{aligned}
$$

Because $\varkappa$ is one dimensional, it is straightforward to compute A_n and B_n explicitly, leading to

$$
A_n=\begin{cases}0 & \text{if } n_2=0\,,\\ d-2\alpha(\dot{\rho}) & \text{if } n_2=1\,,\end{cases}
$$

and

$$
B_n=\begin{cases}0 & \text{if } n_3=0\,,\\ 2d-3\beta_1(\dot{\rho})-3\beta_2(\dot{\rho}) & \text{if } n_3=1\,,\\ d-3\beta_2(\dot{\rho}) & \text{if } n_3=2\,.\end{cases}
$$

Finally, combining the above with Lemma 4, one arrives at

$$|\{i\,|\,k_i\in 3\mathbb{Z}+1\}| = \sum_i (\mathfrak{s}_3(1,\text{-}k_i)-\mathfrak{s}_3(2,\text{-}k_i)) = \frac{1}{3}\,(d-B_1+B_2)=\beta_1\,,$$

$$|\{i\,|\,k_i\in 3\mathbb{Z}+2\}| = \sum_i (\mathfrak{s}_3(2,\text{-}k_i)-\mathfrak{s}_3(3,\text{-}k_i)) = \frac{1}{3}\,(d-B_2+B_3)=\beta_2\,,$$

$$|\{i\,|\,k_i\in 2\mathbb{Z}+1\}| = \sum_i (\mathfrak{s}_2(1,\text{-}k_i)-\mathfrak{s}_2(2,\text{-}k_i)) = \frac{1}{2}\,(d-A_1+A_2)=\alpha\,.$$

□

Lemma 7. *If $F_1,\ldots,F_d\in\mathsf{M}(\rho)$ generate freely the module $\mathsf{M}(\rho)$, then*

$$F_1\wedge F_2\wedge\ldots\wedge F_d = K\,\eth(\tau)^{\sum_i w_i}\,. \tag{26}$$

Proof. It follows from Eqs.(24) and (20) that

$$F_1\wedge F_2\wedge\ldots\wedge F_d = K\frac{\Delta_\epsilon(\rho)}{\prod_i \mathfrak{f}_{-k_i}} =$$

$$= \frac{K\,\eth(\tau)^{d\epsilon}}{\prod_i \mathfrak{f}_{-k_i}}\left(\frac{E_4(\tau)}{\eth(\tau)^4}\right)^{\beta_1(\dot\rho)+2\beta_2(\dot\rho)}\left(\frac{E_6(\tau)}{\eth(\tau)^6}\right)^{\alpha(\dot\rho)}\,.$$

On the other hand,

$$\prod_i \mathfrak{f}_{-k_i} = E_4(\tau)^{\sum_i (\text{-}k_i)_3}\,E_6(\tau)^{\sum_i (\text{-}k_i)_2}\,\Delta(\tau)^{\sum_i (\text{-}k_i)_\infty}\,.$$

But, according to Eq.(25),

$$\sum_i (\text{-}k_i)_3 = \beta_1(\dot\rho)+2\beta_2(\dot\rho)\,,$$

$$\sum_i (\text{-}k_i)_2 = \alpha(\dot\rho)\,,$$

$$\sum_i (\text{-}k_i)_\infty = -\frac{1}{6}\sum_i k_i - \frac{\alpha(\dot\rho)}{2} - \frac{\beta_1(\dot\rho)+2\beta_2(\dot\rho)}{3}\,,$$

leading to

$$F_1\wedge F_2\wedge\ldots\wedge F_d = K\,\eth(\tau)^{d\epsilon+2\sum_i k_i}\,.$$

□

Corollary. *The sum of the fundamental weights cannot be negative, $\sum_i w_i \geq 0$.*

Proof. Should $\sum_i w_i$ be negative, the exterior product $F_1\wedge F_2\wedge\ldots\wedge F_d$ would not be holomorphic. □

We note that Lemma 7 and its corollary hold more generally for any set of forms that generate freely a submodule $\mathsf{m}<\mathsf{M}(\rho)$ that is D-stable, i.e. for which $\mathsf{D}\mathsf{m}<\mathsf{m}$, where D denotes the covariant derivative.

Theorem 3. *Let $P_\rho(z)=\sum_i z^{w_i}$ denote the Hilbert-polynomial of* $\mathsf{M}(\rho)$*, and $\zeta=e^{\frac{2\pi\mathrm{i}}{3}}$ a primitive third root of unity. Then*

$$P_\rho(-\mathrm{i}) = \mathrm{Tr}\,\rho\begin{pmatrix} 0 & \text{-}1 \\ 1 & 0 \end{pmatrix} \tag{27}$$

and

$$P_\rho\left(\zeta^{\pm 1}\right) = \mathrm{Tr}\,\rho\begin{pmatrix} 0 & \text{-}1 \\ 1 & \text{-}1 \end{pmatrix}^{\mp 1} . \tag{28}$$

Proof. Clearly, since $w_i=2k_i+\epsilon$, one has

$$P_\rho(-\mathrm{i})=(-\mathrm{i})^{\epsilon}\sum_{i=1}^{d}(-1)^{k_i} .$$

But it follows from Eq.(25) that

$$\begin{aligned}\sum_{i=1}^{d}(-1)^{k_i} &= \left|\{i\,|\,k_i\in 2\mathbb{Z}\}\right| - \left|\{i\,|\,k_i\in 2\mathbb{Z}+1\}\right| = d-2\alpha(\dot\rho)\\ &= \mathrm{Tr}\,\dot\rho\begin{pmatrix} 0 & \text{-}1 \\ 1 & 0 \end{pmatrix} = \varkappa\begin{pmatrix} 0 & \text{-}1 \\ 1 & 0 \end{pmatrix}^{\text{-}\epsilon}\mathrm{Tr}\,\rho\begin{pmatrix} 0 & \text{-}1 \\ 1 & 0 \end{pmatrix} = \mathrm{i}^{\epsilon}\,\mathrm{Tr}\,\rho\begin{pmatrix} 0 & \text{-}1 \\ 1 & 0 \end{pmatrix} .\end{aligned}$$

Similarly,

$$P_\rho\left(\zeta^{\pm 1}\right)=\zeta^{\pm\epsilon}\sum_{i=1}^{d}\zeta^{\pm 2k_i} .$$

But

$$\begin{aligned}\sum_{i=1}^{d}\zeta^{\pm 2k_i} &= \left|\{i\,|\,k_i\in 3\mathbb{Z}\}\right| + \zeta^{\pm 2}\left|\{i\,|\,k_i\in 3\mathbb{Z}+1\}\right| + \zeta^{\pm 4}\left|\{i\,|\,k_i\in 3\mathbb{Z}+2\}\right|\\ &= d-\beta_1(\dot\rho)-\beta_2(\dot\rho)+\beta_1(\dot\rho)\,e^{\mp\frac{2\pi\mathrm{i}}{3}}+\beta_2(\dot\rho)\,e^{\pm\frac{2\pi\mathrm{i}}{3}} = \mathrm{Tr}\,\dot\rho\begin{pmatrix} 0 & \text{-}1 \\ 1 & \text{-}1 \end{pmatrix}^{\mp 1}\\ &= \varkappa\begin{pmatrix} 0 & \text{-}1 \\ 1 & \text{-}1 \end{pmatrix}^{\pm\epsilon}\mathrm{Tr}\,\rho\begin{pmatrix} 0 & \text{-}1 \\ 1 & \text{-}1 \end{pmatrix}^{\mp 1} = \zeta^{\pm 2\epsilon}\,\mathrm{Tr}\,\rho\begin{pmatrix} 0 & \text{-}1 \\ 1 & \text{-}1 \end{pmatrix}^{\mp 1} ,\end{aligned}$$

proving the assertion. □

5. Summary and outlook

We have been investigating the distribution of fundamental weights (i.e. the weights of a freely generating set) of the module $\mathsf{M}(\rho)$ of holomorphic vector-valued modular forms for well behaved representations $\rho\colon \mathbf{\Gamma}\to \mathrm{GL}(V)$. We established relations between the weight distribution and properties of the representation through the consideration of freely generating sets for the modules of weakly holomorphic forms of diverse weights, leading to the trace formulas Eqs.(27) and (28), expressing the value of the Hilbert-polynomial $P_\rho(z)$ evaluated at special arguments in terms of traces of representation matrices. These results could lead to a better understanding of the fundamental weights, and to effective procedures for determining them from representation theoretic data.

Of course, knowing the fundamental weights is far from knowing explicitly a freely generating set, but this restricted information could already be very useful. In this respect, the connection between holomorphic generators and weakly holomorphic ones of weight 0, as expressed by Theorem 1, seems to be crucial: indeed, for simple examples of low dimension, such considerations are already enough to compute the q-expansions of a freely generating set. Promoting these *ad hoc* methods to some generally valid algorithm would be especially important for the study and applications of vector-valued modular forms.

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